Some estimates of the coefficients of polynomials

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Abstract

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, then Rahman and Schmeisser [4] proved that for every $p \in [0, \infty]$ the inequality

$$|a_n| + |a_0| \leq 2\frac{\|P\|_p}{\|1+z\|_p}$$

holds, where

$$\|P\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \quad (0 < p < \infty)$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|$$

and

$$\|P\|_0 := exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} log|P(e^{i\theta})| d\theta \right\}.$$

In this paper, we obtain some estimates of the coefficients of a polynomial $P(z)$ which among other things include the above inequality as a special case.

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1 Introduction and statement of results

Let $\mathcal{P}_n$ denotes the class of all polynomials of degree at most n with complex coefficients.. For $P \in \mathcal{P}_n$ define,

$$\|P\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \quad (0 < p < \infty)$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|$$

and

$$\|P\|_0 := exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} log|P(e^{i\theta}|d\theta \right\}$$

If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then by Rouche's theorem, it follows that

$$P(z) + \mu\|P\|_\infty = a_n z^n + a_{n-1} z^{n-1} + ... + (a_0 + \mu\|P\|_\infty)$$

does not vanish in the unit disk $|z| < 1$ for any choice of $\mu \in \mathbb{C}$ with $|\mu| = 1$. It follows that $|a_0 + \mu\|P\|_\infty| \geq |a_n|$ for each $\mu \in \mathbb{C}$ with $|\mu| = 1$. By choosing the argument of μ suitably, we get

$$|a_n| + |a_0| \leq \|P\|_\infty \tag{1.1}$$

This inequality is a well known result called as Visser's inequality [5]. Equality in (1.1) holds only when $a_j = 0$ for $j = 1, 2, ...n - 1$.
Different variants of this inequality can be found in [3]. Rahman and Schmeisser [?] extended the inequality (1.1) to L_p norms and proved the following:
If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then

$$|a_n| + |a_0| \leq 2\frac{\|P\|}{\|1+z\|_p} \qquad for\ each\ \ p \in [0, \infty] \tag{1.2}$$

In this paper, we first prove the following result which among other things include inequalities (1.1) and (1.2) as special cases. In fact, we prove the following theorem.
Theorem 1.1. Let $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then for each $0 \leq p < \infty$,

$$|a_0| + \frac{|a_k|}{\binom{n}{k}} \leq 2\frac{\|P\|_p}{\|1+z\|_p} \qquad for\ each\ \ 0 \leq p < \infty \tag{1.3}$$

where $k = 1, 2, ...n$.
For $k = n$, the inequality (1.3) reduces to (1.2).
If we let $p \to \infty$ in (1.3), we obtain the following result, from which the Visser's inequality follows when $k = n$.
Corollary 1.2. Let $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$, then

$$|a_0| + \frac{|a_k|}{\binom{n}{k}} \leq \|P\|_\infty$$

where $k = 1, 2, ...n$.
Theorem 1.1 can be improved if we restrict ourselves to the class of polynomials having all zeros in $|z| \leq 1$. In this direction, we prove:

Theorem 1.3. If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$ having all its zeros in $|z| \leq 1$, then for each $0 \leq p < \infty$,

$$|a_0| + \frac{|a_{n-k}|}{\binom{n}{k}} \leq 2M_p \frac{\|P\|_p}{\|1+z\|_p}$$

$$where\ k = 1, 2, ...n\ and\ M_p = \begin{cases} 1 & if\ k = n. \\ \dfrac{1}{\|1+z\|_p} & if\ k < n. \end{cases}$$

Note that $0 < M_p < 1$ for $k < n$ and $p > 0$.

2 Lemmas

We first describe a result of Arestov [1]
For $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \cdots, \gamma_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum\limits_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, we define
$C_{\boldsymbol{\gamma}} P(z) = \sum\limits_{j=0}^{n} \gamma_j a_j z^j$
The operator $C_{\boldsymbol{\gamma}}$ is said to be admissible if it preserves one of the following properties:
(i) $P(z)$ has all its zeros in $|z| \leq 1$
(ii) $P(z)$ has all its zeros in $|z| \geq 1$

The result of Arestov [1], (Theorem 2) may now be stated as follows,

Lemma 2.1. Let $\varphi(x) = \psi(log x)$, where ψ is a convex non-decreasing function on $\mathbb{R}$. Then for all $P \in \mathcal{P}_n$ each admissible operator $C_{\boldsymbol{\gamma}}$,

$$\int_0^{2\pi} \varphi\big(|C_{\boldsymbol{\gamma}} P(e^{i\theta})|\big) d\theta \le \int_0^{2\pi} \varphi\big(|c(\boldsymbol{\gamma}) P(e^{i\theta})|\big) d\theta$$

where $c(\boldsymbol{\gamma}) = \max(|\gamma_0|, |\gamma_n|)$

In particular Lemma 2.1 applies with $\varphi : x \to x^p$ for every $p \in (0, \infty)$ and with $\varphi : x \to logx$ as well. Therefore, we have for $0 \le p < \infty$.

$$\left\{ \int_0^{2\pi} (|C_{\boldsymbol{\gamma}} P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \le c(\boldsymbol{\gamma}) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \tag{2.1}$$

Lemma 2.2. Let $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} a_j z^j$ having all its zeros in

$|z| \le 1$, then for $k = 1, 2, ..., n$, ϕ real and each $p > 0$

$$\int_0^{2\pi} \left| \left(\overline{a}_0 e^{in\theta} + \frac{\overline{a}_{n-k}}{\binom{n}{k}} e^{ik\theta} \right) e^{i\phi} + \left(\frac{\overline{a}_k}{\binom{n}{k}} e^{i(n-k)\theta} + \overline{a}_n \right) \right| d\theta \le \Omega^p \int_0^{2\pi} |P^*(e^{i\theta})|^p d\theta$$

$$where\ k = 1, 2, ...n\ and\ \Omega = \begin{cases} 1 & if\ k < n. \\ |1 + e^{i\phi}| & if\ k = n. \end{cases}$$

Proof: Since P(z) has all its zeros in $|z| \le 1$, then all zeros of $P^*(z) = z^n \overline{P(1/\overline{z})}$ lie in $|z| > 1$ and $|P(z)| = |P^*(z)|$ for $|z| = 1$. Therefore, $\dfrac{P(z)}{P^*(z)}$ is analytic in $|z| \le 1$.

By maximum modulus principle, we have

$$|P(z)| \le |P^*(z)| \ \ for \ \ |z| \le 1.$$

or equivalently,

$$|P^*(z)| \le |P(z)| \ \ for \ \ |z| \ge 1.$$

By Rouche's theorem , all the zeros of the polynomial

$$P^*(z) - \mu P(z) = \sum_{j=0}^{n} (\overline{a}_{n-j} - \mu a_j) z^j$$

lie in $|z| \le 1$ for every $\mu \in \mathbb{C}$ with $|\mu| > 1$. If $z_1, z_2, \cdots, z_n$ are roots of $P^*(z) - \mu P(z)$, then $|z_j| \le 1$, $j = 1, 2, \cdots, n$ and we have by Viete's formula for $k = 1, 2, \cdots, n$,

$$(-1)^{n-k}\left(\frac{\overline{a}_{n-k}+\mu a_k}{\overline{a}_0+\mu a_n}\right)=\sum_{1\leq i_1<i_2<\ldots<i_{n-k}\leq n} z_{i_1}z_{i_2}\cdots z_{i_{n-k}}$$

This gives

$$\left|\frac{\overline{a}_{n-k}+\mu a_k}{\overline{a}_0+\mu a_n}\right|=\sum_{1\leq i_1<i_2<\ldots<i_{n-k}\leq n}\left|z_{i_1}z_{i_2}\cdots z_{i_{n-k}}\right|\leq\binom{n}{n-k}=\binom{n}{k} \quad (2.2)$$

Therefore, all the zeros of the polynomial

$$\begin{aligned}\mathcal{G}(z)&=(\overline{a}_0+\mu a_n)z^n+\frac{\overline{a}_{n-k}+\mu a_k}{\binom{n}{k}}z^k\\&=\overline{a}_0z^n+\frac{\overline{a}_{n-k}}{\binom{n}{k}}z^k+\mu\left(a_nz^n+\frac{a_k}{\binom{n}{k}}z^k\right)\end{aligned}$$

lie in $|z|\leq 1$ for $\mu\in\mathbb{C}$ with $|\mu|>1$. So that if $s>1$, the polynomial

$$\mathcal{G}(sz)=\overline{a}_0(sz)^n+\frac{\overline{a}_{n-k}}{\binom{n}{k}}(sz)^k+\mu\left(a_n(sz)^n+\frac{a_k}{\binom{n}{k}}(sz)^k\right)$$

has all its zeros in $|z|<1$. This gives

$$\left|\overline{a}_0(sz)^n+\frac{\overline{a}_{n-k}}{\binom{n}{k}}(sz)^k\right|\leq\left|a_n(sz)^n+\frac{a_k}{\binom{n}{k}}(sz)^k\right| \quad (2.3)$$

for $|z|\geq 1$. For if inequality (2.3) is not true, then there exists a point w with $|w|\geq 1$ such that

$$\left|\overline{a}_0(sw)^n+\frac{\overline{a}_{n-k}}{\binom{n}{k}}(sw)^k\right|>\left|a_n(sw)^n+\frac{a_k}{\binom{n}{k}}(sw)^k\right|$$

Since all the zeros of P(z) lie in $|z|\leq 1$, then by similar argument as in (2.2), we have $|a_n|\geq\left|\frac{a_k}{\binom{n}{k}}\right|$ which implies that $a_n(sw)^n\geq\frac{a_k}{\binom{n}{k}}(sw)^k\neq 0$. If we take

$$\mu=-\frac{\overline{a}_0(sw)^n+\frac{\overline{a}_{n-k}}{\binom{n}{k}}(sw)^k}{a_n(sw)^n+\frac{a_k}{\binom{n}{k}}(sw)^k}$$

then μ is a well defined complex number with $|\mu| > 1$ and with this choice of μ we obtain $\mathcal{Q}(sw) = 0$ where $|w| \geq 1$ which contradicts the fact that all the zeros of $\mathcal{Q}(sz)$ lie in $|z| < 1$. Thus (2.3) holds. If we let $s \to 1$ in (2.3) and using continuity, it follows that,

$$\left|\overline{a}_0 z^n + \frac{\overline{a}_{n-k}}{\binom{n}{k}} z^k\right| \leq \left|a_n z^n + \frac{a_k}{\binom{n}{k}} z^k\right| = \left|\overline{a}_n + \frac{\overline{a}_k}{\binom{n}{k}} z^{n-k}\right| \tag{2.4}$$

for $|z| = 1$.Again, since $|a_n| \geq \left|\frac{a_k}{\binom{n}{k}}\right|$ the polynomial $\left|\overline{a}_n + \frac{\overline{a}_k}{\binom{n}{k}} z^{n-k}\right|$ does not vanish in $|z| < 1$.By the maximum modulus principle, it follows that,

$$\left|\overline{a}_0 z^n + \frac{\overline{a}_{n-k}}{\binom{n}{k}} z^k\right| < \left|\overline{a}_n + \frac{\overline{a}_k}{\binom{n}{k}} z^{n-k}\right| \quad for \ \ |z| < 1.$$

By Rouche's theorem, the polynomial

$$C_\gamma P^*(z) = \left(\overline{a}_0 z^n + \frac{\overline{a}_{n-k}}{\binom{n}{k}} z^k\right) e^{i\phi} + \left(\frac{\overline{a}_k}{\binom{n}{k}} z^{n-k} + \overline{a}_n\right)$$

has all its zeros in $|z| \geq 1$. Therefore , C_γ is an admissible operator. Applying (2.1) of lemma (2.1), the required result follows for $p > 0$.

Lemma 2.3. Let α be a complex number independent of θ, where θ is real. Then for each $p > 0$

$$\int_0^{2\pi} \left|\alpha + e^{i\theta}\right|^p d\theta = \int_0^{2\pi} \left|1 + |\alpha| e^{i\theta}\right|^p d\theta$$

Lemma 2.4. Let n be a positive integer and $0 \leq p \leq \infty$
$\|1 + z^n\|_p = \|1 + z\|_p$.

For the above two Lemmas 2.3 and 2.4 see [2].

3 Proof of theorems

Proof of theorem 1.1. By hypothesis $P \in \mathcal{P}_n$, we can write

$P(z) = P_1(z) P_2(z)$

where all the zeros of $P_1(z)$ lie in $|z| > 1$ and all the zeros of $P_2(z)$ lie in $|z| \leq 1$.First, we suppose that $P_2(z)$ has no zero on $|z| = 1$.Let the degree of polynomial $P_1(z)$ be k , then the polynomial $P_1^*(z)$ has all its zeros in $|z| \leq 1$ and $|P_1^*(z)| = |P_1(z)|$ for $|z| = 1$
.

Consider the polynomial

$F(z) = P_1^*(z)P_2(z)$

then all the zeros of F(z) lie in $|z| \leq 1$

$$\begin{aligned} |F(z)| &= |P_1^*(z)||P_2(z)| \\ &= |P_1(z)||P_2(z)| \\ &= |P(z)| \end{aligned}$$

By the maximum modulus principle, it follows that

$|P(z)| \leq |F(z)|$ for $|z| \geq 1$

Since $F(z) \neq 0$ for $|z| \geq 1$, a direct application of Rouche's theorem shows that the polynomial $H(z) = P(z) + \mu F(z)$ has all its zeros in $|z| < 1$, for every $\mu \in \mathbb{C}$ with $|\mu| > 1$.Let $F(z) = \sum_{j=0}^{n} b_j z^j$, then the polynomial

$$H(z) = \sum_{j=0}^{n} (a_j + \mu b_j) z^j$$

has all its zeros in $|z| < 1$. If $w_1, w_2, \cdots, w_n$ be roots of H(z), then $|w_j| < 1, j = 1, 2, ..., n$ and we have by Viete's formula for $k = 1, 2, ..., n$

$$(-1)^{n-k} \left(\frac{a_k + \mu b_k}{a_0 + \mu b_0} \right) = \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n} w_{i_1} w_{i_2} \cdots w_{i_{n-k}}$$

Now, proceeding similarly as in the proof of Lemma 2.2, we obtain

$$\left| a_0 z^n + \frac{a_k}{\binom{n}{k}} z^k \right| \leq \left| b_0 z^n + \frac{b_k}{\binom{n}{k}} z^k \right|$$

$for\ |z| \leq 1$. This implies for each $p > 0$ and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi} \left| a_0 e^{in\theta} + \frac{a_k}{\binom{n}{k}} e^{ik\theta} \right|^p d\theta \leq \int_0^{2\pi} \left| b_0 e^{in\theta} + \frac{b_k}{\binom{n}{k}} e^{ik\theta} \right|^p d\theta \qquad (3.1)$$

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Again, since all the zeros of $F(z) = \sum_{j=0}^{n} b_j z^j$ lie in $|z| < 1$, similarly as shown before, the polynomial $b_0 z^n + \frac{b_k}{\binom{n}{k}} z^k$ also has all its zeros in $|z| < 1$.
Therefore the operator $C_{\boldsymbol{\gamma}}$ defined by

$$C_{\boldsymbol{\gamma}} F(z) = b_0 z^n + \frac{b_k}{\binom{n}{k}} z^k$$

is admissible. Hence by (2.1) of Lemma 2.1, for each $p > 0$ we have

$$\int_0^{2\pi} \left| b_0 e^{in\theta} + \frac{b_k}{\binom{n}{k}} e^{ik\theta} \right|^p d\theta \le (c(\boldsymbol{\gamma}))^p \left(\int_0^{2\pi} |F(e^{i\theta})|^p d\theta \right) \tag{3.2}$$

where $c(\boldsymbol{\gamma}) = \max(|\gamma_0|, |\gamma_n|) = 1$. Combining inequalities (3.1),(3.2) and noting that $|F(e^{i\theta})| = |P(e^{i\theta})|$, we obtain for each $p > 0$.

$$\left\{ \int_0^{2\pi} \left| a_0 e^{in\theta} + \frac{a_k}{\binom{n}{k}} e^{ik\theta} \right|^p d\theta \right\}^{1/p} \le \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p} \tag{3.3}$$

Here, we claim that for n and k being positive with $n > k$, we have,

$$\left\| \overline{a}_0 z^n + \frac{\overline{a}_{n-k}}{\binom{n}{k}} z^k \right\|_p \ge \frac{|\overline{a}_0| + \frac{|\overline{a}_{n-k}|}{\binom{n}{k}}}{2} \|1 + z\|_p. \tag{3.4}$$

Proof of the claim. If $|\overline{a}_0| = 0$ then (3.4) follows by fact $\|1 + z\|_p < 2$. Thus we assume that if $|\overline{a}_0| \neq 0$, then by Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned} \left\| \overline{a}_0 z^n + \frac{\overline{a}_{n-k}}{\binom{n}{k}} z^k \right\|_p &= |\overline{a}_0| \left\| z + \frac{\frac{\overline{a}_{n-k}}{\binom{n}{k}}}{\overline{a}_0} \right\|_p \\ &= |\overline{a}_0| \left\| 1 + \frac{\left| \frac{\overline{a}_{n-k}}{\binom{n}{k}} \right|}{\overline{a}_0} z \right\|_p . \end{aligned}$$

From the inequality

$$\left|\frac{1+re^{i\theta}}{1+e^{i\theta}}\right| \geq \frac{1+r}{2} \text{ with } r = \left|\frac{\frac{\overline{a}_{n-k}}{\binom{n}{k}}}{\overline{a}_0}\right| \text{ and } 0 \leq \theta < 2\pi$$

We deduce,

$$|\overline{a}_0|\left|1+\left|\frac{\frac{\overline{a}_{n-k}}{\binom{n}{k}}}{\overline{a}_0}\right|e^{i\theta}\right| \geq \frac{|\overline{a}_0| + \frac{|\overline{a}_{n-k}|}{\binom{n}{k}}}{2}\left|1+e^{i\theta}\right|$$

This implies,

$$|\overline{a}_0|\left\|1+\left|\frac{\frac{\overline{a}_{n-k}}{\binom{n}{k}}}{\overline{a}_0}\right|e^{i\theta}\right\|_p \geq \frac{|\overline{a}_0| + \frac{|\overline{a}_{n-k}|}{\binom{n}{k}}}{2}\left\|1+e^{i\theta}\right\|_p$$

Using this in conjunction with (3.5), the desired claim follows.
Now using (3.3) in conjunction with (3.4), we get,

$$|a_0| + \frac{|a_k|}{\binom{n}{k}} \leq 2\frac{\|P\|_p}{\|1+z\|_p} \tag{3.5}$$

In case $P_1(z)$ has a zero on $|z| = 1$, inequality (3.5) follows by continuity. This proves Theorem 1.1 for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$.

Proof of Theorem 1.3. Since $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ has all its zeros in $|z| \leq 1$
Therefore, by (2.4), we have,

$$\left|\overline{a}_0 e^{in\theta} + \frac{\overline{a}_{n-k}}{\binom{n}{k}}e^{ik\theta}\right| \leq \left|\frac{\overline{a}_k}{\binom{n}{k}}e^{i(n-k)\theta} + \overline{a}_n\right| \tag{3.6}$$

for $k = 1, 2, ..., n$

Also, by Lemma 2.2,

$$\int_0^{2\pi}\left|H(\theta) + e^{i\phi}G(\theta)\right|^p d\theta \leq \Omega^p \int_0^{2\pi}|P^*(e^{i\theta})|^p d\theta \tag{3.7}$$

where $H(\theta) = \overline{a}_0 e^{in\theta} + \dfrac{\overline{a}_{n-k}}{\binom{n}{k}}e^{ik\theta}$

and
$G(\theta) = \frac{\overline{a}_k}{\binom{n}{k}} e^{i(n-k)\theta} + \overline{a}_n.$

Integrating both sides of (3.7) with respect to ϕ from 0 to 2π, we get for each $p > 0$ and ϕ real

$$\int_0^{2\pi}\int_0^{2\pi} \left|H(\theta) + e^{i\phi}G(\theta)\right|^p d\theta d\phi \leq \int_0^{2\pi} \Omega^p \int_0^{2\pi} |P^*(e^{i\theta})|^p d\theta d\phi \tag{3.8}$$

Now, for every real ϕ and $t \geq 1$ and $p > 0$, we have

$$\int_0^{2\pi} \left|t + e^{i\phi}\right|^p d\phi \geq \int_0^{2\pi} \left|1 + e^{i\phi}\right|^p d\phi$$

If $H(\theta) \neq 0$, we take $t = |G(\theta)/H(\theta)|$, then by (3.6) $t \geq 1$ and by using Lemma 2.3 we get

$$\begin{aligned}\int_0^{2\pi} \left|H(\theta) + e^{i\phi}G(\theta)\right|^p d\theta &= |H(\theta)|^p \int_0^{2\pi} \left|1 + e^{i\phi}\frac{G(\theta)}{H(\theta)}\right|^p d\phi \\ &= |H(\theta)|^p \int_0^{2\pi} \left|e^{i\phi} + \left|\frac{G(\theta)}{H(\theta)}\right|\right|^p d\phi \\ &\geq |H(\theta)|^p \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi\end{aligned}$$

For $H(\theta) = 0$, this inequality is trivially true. Using this in (3.8),we conclude that, for real ϕ,

$$\int_0^{2\pi} |H(\theta)|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \leq \int_0^{2\pi} \Omega^p d\phi \int_0^{2\pi} |P^*(e^{i\theta})|^p d\theta.$$

which implies,

$$\left\{\int_0^{2\pi} \left|\overline{a}_0 e^{in\theta} + \frac{\overline{a}_{n-k}}{\binom{n}{k}} e^{ik\theta}\right|^p d\theta\right\}^{1/p} \leq \frac{\left\{\int_0^{2\pi} \Omega^p d\phi\right\}^{1/p} \left\{\int_0^{2\pi} \left|P^*(e^{i\theta})\right|^p d\theta\right\}^{1/p}}{\left\{\int_0^{2\pi} |1 + e^{i\phi}|^p d\phi\right\}^{1/p}}$$

Using this in conjunction with (3.4), we get

$$|\overline{a}_0| + \frac{|\overline{a}_{n-k}|}{\binom{n}{k}} \leq 2M_p \frac{\|P^*(z)\|_p}{\|1+z\|_p}$$

$$or \quad |a_0| + \frac{|a_{n-k}|}{\binom{n}{k}} \leq 2M_p \frac{\|P(z)\|_p}{\|1+z\|_p}$$

$$where\ k = 1, 2, ...n\ and\ M_p = \begin{cases} 1 & if\ k = n. \\ \dfrac{1}{\|1+z\|_p} & if\ k < n. \end{cases}$$

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